First order linear systems

**Vocabulary/Concepts:** system of differential equations; linear, time-independent, homogeneous; matrix, matrix multiplication; solution, trajectory, phase portrait; companion matrix.

11. Practice in matrix multiplication: Compute the following products:

\[
\begin{bmatrix} 1 & 2 \\ 1 & 3 \\ x + 2y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 3 \\ y \\ v \end{bmatrix}.
\]

2. Multiplying by a matrix \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) sends a vector \( \begin{bmatrix} x \\ y \end{bmatrix} \) to another vector \( A \begin{bmatrix} x \\ y \end{bmatrix} \). This operation can be visualized by thinking about where it sends the square with corners \( O = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, j = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, i + j = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \).

For each of the following matrices \( A \), draw segments connecting the dots \( 0, Ai, A(i+j), Aj, 0 \), and invent verbal description or name for the operation.

\( A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \)

\( A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \): holding x-direction unchanged, but lengthening y-direction by a factor of 2.

\( A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \): holding the bottom two vertices on x-axis fixed, but moving the upper two vertices horizontally to the right by unit 1.

\( A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \): keeping the dimensions unchanged, but being reflected with respect to x-axis.

\( A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \): keeping the dimensions unchanged, but being reflected with respect to the line \( y = x \).

\( A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \): holding the vertex at the origin fixed, first rotating the square 45 degrees clockwise, then flipping it with respect to x-axis, and finally stretching the four sides to the length of \( \sqrt{2} \).
3. What is the companion matrix $A$ of the second order equation $\ddot{x} + 2\dot{x} + 2x = 0$? Find two independent real solutions of this second order equation. Let $x_1(t)$ denote the solution with initial condition $x_1(0) = 0$, $\dot{x}_1(0) = 1$. Find it, and then write down the corresponding solution $u_1(t) = \begin{bmatrix} x_1(t) \\ \dot{x}_1(t) \end{bmatrix}$ of the equation $\ddot{u} = Au$. What is $u_1(0)$? Sketch the graphs of $x_1(t)$ and of $\dot{x}_1(t)$, and sketch the trajectory of the solution $u_1(t)$. Compare these pictures.

Sketch a few more trajectories to fill out the phase portrait. In particular sketch the trajectory of $u_2(t)$ with $u_2(0) = i$.

When trajectories of this companion equation cross the $x$ axis, at what angle do they cross it?

The companion matrix is $\begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$. The characteristic polynomial is $p(s) = s^2 + 2s + 2$ with roots $s = -1 \pm i$, so two independent complex solutions are $e^{(-1+i)t}$ and $e^{(-1-i)t}$. We can combine them to form independent real solutions $e^{-t} \cos t$ and $e^{-t} \sin t$. Considering the given initial condition, we choose $x_1(t) = e^{-t} \sin t$, so $u_1(t) = \begin{bmatrix} e^{-t} \sin t \\ e^{-t}(\cos t - \sin t) \end{bmatrix}$.

$x_1$ has envelope $\pm e^{-t}$, which decays exponentially. The graph of $x_1$ oscillates inside the envelope, and it touches the envelope at odd multiples of $\pi/2$. $\dot{x}_1$ has envelope $\pm \sqrt{2}e^t$, and the graph touches the envelope when $t$ has the form $\frac{4k}{4} - \pi$. The trajectory is an inward spiral, elongated in the northwest-southeast direction. When trajectories cross the $x$ axis, they cross at an angle of $\pi/2$.

4. Let $a + bi$ be a general complex number. There is a matrix $A$ such that if $(a + bi)(x + yi) = (v + wi)$ then

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} v \\ w \end{bmatrix}$$

Find it. What is it for $a + bi = 2$? For $a + bi = i$? For $a + bi = 1 + i$? Draw the parallelograms discussed in (2) for these matrices.

We have $v = ax - by$, $w = ay + bx$, so $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. For $a + bi = 2$, $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, and the parallelogram is a square of length 2. For $a + bi = i$, $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and the parallelogram is a square of length 1, rotated by 90 degrees counterclockwise around the origin. For $a + bi = 1 + i$, $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, and the parallelogram is a square of length $\sqrt{2}$, rotated by 45 degrees counterclockwise around the origin.