Laplace transform

1. Find (from the rules and formulas) the Laplace transform of $u(t)e^{-t}(t^2 + 1)$. Here we use the linearity and $s$-shift rule of the Laplace transform, as well as the formula $\mathcal{L}[e^{-t}] = \frac{1}{s+1}$ and $\mathcal{L}[t^2] = \frac{2}{s^3}$. So the result is given by

$$
\mathcal{L}[e^{-t}(t^2 + 1)] = \mathcal{L}[e^{-t}t^2] + \mathcal{L}[e^{-t}]
$$

$$
= \frac{2}{(s+1)^3} + \frac{1}{s+1}
$$

$$
= \frac{s^2 + 2s + 3}{(s+1)^3}.
$$

2. Let $f(t) = e^{-t}\cos(3t)$. From the rules and tables, what is $F(s) = \mathcal{L}[f(t)]$? Compute the generalized derivative $f'(t)$ and its Laplace transform. Verify the $t$-derivative rule in this case.

We can read directly from the table that $\mathcal{L}[\cos(3t)] = \frac{s}{s^2 + 9}$. Hence by the $s$-shift rule of the Laplace transform,

$$
F(s) = \mathcal{L}[e^{-t}\cos(3t)]
$$

$$
= \mathcal{L}[\cos(3t)](s + 1)
$$

$$
= \frac{s + 1}{(s + 1)^2 + 9}
$$

$$
= \frac{s + 1}{s^2 + 2s + 10}.
$$

When $t > 0$, $f'(t) = -e^{-t}(\cos(3t) + 3\sin(3t))$. Since $f(t)$ is actually $f(t)u(t)$, and $f(0+) = 1$, so

$$
f'(t) = \delta(t) - u(t)e^{-t}(\cos(3t) + 3\sin(3t)).
$$

Again from the table, we know $\mathcal{L}[\sin(3t)] = \frac{3}{s^2 + 9}$ and $\mathcal{L}[\delta(t)] = 1$. Therefore, by the linearity and the $s$-shift rule,

$$
\mathcal{L}[f'(t)] = \mathcal{L}[\delta(t)] - \mathcal{L}[e^{-t}\cos(3t)] - 3\mathcal{L}[e^{-t}\sin(3t)]
$$

$$
= 1 - \frac{s + 1}{s^2 + 2s + 10} - 3\frac{3}{(s + 1)^2 + 9}
$$

$$
= \frac{s^2 + s}{s^2 + 2s + 10}.
$$

Hence $\mathcal{L}[f'(t)] = sF(s)$, which verifies the $t$-derivative rule of the Laplace transform.
3. Find the inverse Laplace transform for each of the following.

\[ \begin{align*}
\frac{2s + 1}{s^2 + 9} &= \frac{2}{s^2(s - 1)}, \\
\frac{s^2 + 2}{s^3 - s} &= \frac{2}{s^2(s - 1)}
\end{align*} \]

Using linearity and the cosine and sine formulas, we find that \( \mathcal{L}[a \sin(3t) + b \cos(3t)] = \frac{3a + bs}{s^2 + 9} \). So set \( a = 1/3 \) and \( b = 2 \), we have \( \mathcal{L}^{-1}[\frac{3s + 1}{s^2 + 9}] = \frac{1}{3} \sin(3t) + 2 \cos(3t) \).

For the second one, since the denominator is \( s^3 - s = s(s - 1)(s + 1) \), we expect a combination of 1, \( e^t \), and \( e^{-t} \). For constants \( a, b \) and \( c \), we have

\[ \mathcal{L}[ae^{-t} + b + ce^t] = \frac{a}{s + 1} + \frac{b}{s} + \frac{c}{s - 1} \]

\[ = \frac{a(s^2 - s) + b(s - 1) + c(s^2 + s)}{s^3 - s} \]

\[ = \frac{(a + b + c)s^2 + (c - a)s - b}{s^3 - s} \]

So set \( b = -2 \), and \( a = c = 3/2 \), \( \mathcal{L}^{-1}[\frac{s^2 + 2}{s^3 - s}] = \frac{3}{2}(e^{-t} + e^t) - 2 \).

Similarly for the last one, the denominator is \( s^2(s - 1) \), so we are looking for constants \( a, b, c \) such that

\[ \frac{2}{s^2(s - 1)} = \mathcal{L}[a + bt + ce^t] \]

\[ = \frac{a}{s + 1} + \frac{b}{s^2} + \frac{c}{s - 1} \]

\[ = \frac{a(s^2 - s) + b(s - 1) + cs^2}{s^2(s - 1)} \]

\[ = \frac{(a + c)s^2 + (b - a)s - b}{s^2(s - 1)} \]

So set \( b = -2 \), and \( a = -c = -2 \), then \( \mathcal{L}^{-1}[\frac{2}{s^2(s - 1)}] = -2(1 + t - e^t) \).

4. Find the unit step and impulse response for the operator \( D + 2I \), using the Laplace transform.

We want to find a solution to \( \dot{x} + 2x = u(t) \), so we take the Laplace transform of both sides. Denote the Laplace transform of \( x(t) \) by \( \mathcal{X}(s) \). By the \( t \)-derivative rule and linearity, \( \mathcal{L}[\dot{x} + 2x] = s\mathcal{X}(s) + 2\mathcal{X}(s) \), and on the right, \( \mathcal{L}[u(t)] = \frac{1}{s} \). We find that \( \mathcal{X}(s) = \frac{1}{s + 2} = \frac{1}{s} + \frac{-1}{s + 2} \). Taking inverse transforms, we find that the unit step response is \( x = \frac{u(t)}{2}(1 - e^{-2t}) \).

We do the same thing to the left side of the equation \( \dot{x} + 2s = \delta(t) \) as above, but now the Laplace transform of the right side is 1, so \( \mathcal{X}(s) = \frac{1}{s + 2} \). We find that unit impulse response is \( x = u(t)e^{-2t} \).

5. Solve \( \dot{x} + 2x = t^2 \) with initial condition \( x(0+) = 1 \), using Laplace transform.

Since the equation is first order, and the initial condition starts at one, we actually want to take the Laplace transform of both sides of a slightly altered equation: \( \dot{x} + 2x = t^2 \).
$2x = t^2 + \delta(t)$. This is because the standard assumption of rest initial conditions requires $x$ to have a jump discontinuity: $x(0^-) = 0$ while $x(0^+) = 1$. This forces $\dot{x}$ to have a $\delta(t)$ term. (An easier way would be to use the $t$-derivative formula for ordinary derivatives.) We find that $(s+2)x(s) = \frac{2}{s^2} + 1$, so we need to find the inverse Laplace transform of $\frac{s^2 + 1}{s^3(s+2)}$. Using partial fractions, we set

$$
\frac{s^3 + 2}{s^3(s+2)} = \frac{a}{s^3} + \frac{b}{s^2} + \frac{c}{s} + \frac{d}{s+2} = \frac{a(s+2) + b(s^2 + 2s) + c(s^3 + 2s^2) + ds^3}{s^3(s+2)} = \frac{(c + d)s^3 + (b + 2c)s^2 + (a + 2b)s + 2a}{s^3(s+2)}
$$

so $a = 1$, $b = -1/2$, $c = 1/4$, and $d = 3/4$. Taking the inverse Laplace transform, we find that $x = \frac{1}{4}u(t)(2t^2 - 2t + 1 + 3e^{-2t})$. 