Recitation 14, March 30, 2010

Fourier Series

Suppose that \( f(t) \) a periodic function and that \( 2\pi \) is a period (so \( f(t + 2\pi) = f(t) \)). (To be honest we also assume that \( f(t) \) is piecewise continuous and that \( f(a) = \frac{1}{2}(f(a-) + f(a+)) \) at points of discontinuity.) Then there is exactly one sequence of numbers \( a_0, a_1, \ldots, b_1, \ldots, \) for which

\[
f(t) = \frac{a_0}{2} + a_1 \cos(t) + a_2 \cos(2t) + \cdots + b_1 \sin(t) + b_2 \sin(2t) + \cdots
\]

The Fourier coefficients are defined as the numbers fitting into this expression. They can be calculated using the integral formulas

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) \, dt, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) \, dt
\]

but often they can be found more easily than this, starting from some basic examples. One basic example is the standard squarewave: \( sq(t) \) is the odd function of period \( 2\pi \) such that \( sq(t) = 1 \) for \( 0 < t < \pi \).

\[
sq(t) = \frac{4}{\pi} \left( \sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \cdots \right) = \frac{4}{\pi} \sum_{k \text{ odd}} \frac{\sin(kt)}{k}
\]

1. Graph the function \( f(t) \) which is even, periodic of period \( 2\pi \), and such that \( f(t) = 2 \) for \( 0 < t < \frac{\pi}{2} \) and \( f(t) = 0 \) for \( \frac{\pi}{2} < t < \pi \). Find its Fourier series in two ways:

   (a) Use the integral expressions for the Fourier coefficients. (Is the function even or odd? What can you say right off about the coefficients?)

   (b) Express \( f(t) \) in terms of \( sq(t) \), substitute the Fourier series for \( sq(t) \), and use some trig id.

   (c) Now find the Fourier series for \( f(t) - 1 \).

2. What is the Fourier series for \( \sin^2 t \)?

3. Graph the odd function \( g(x) \) which is periodic of period \( \pi \) and such that \( g(x) = 1 \) for \( 0 < x < \frac{\pi}{2} \). \( 2\pi \) is also a period of \( g(x) \), so it has a Fourier series as above. Find it by expressing \( g(x) \) in terms of the standard squarewave.

4. Graph the function \( h(t) \) which is odd and periodic of period \( 2\pi \) and such that \( h(t) = t \) for \( 0 < t < \frac{\pi}{2} \) and \( h(t) = \pi - t \) for \( \frac{\pi}{2} < t < \pi \). Find its Fourier series, starting with your solution to 1(c).
5. Explain why any function $g(x)$ is a sum of an even function and an odd function in just one way. What is the even part of $e^x$? What is the odd part?