18.03 Class 21, March 29

Fourier series II

[1] Review
[2] Square wave
[3] Piecewise continuity

[1] Recall from before break: A function $f(t)$ is periodic of period $2L$ if $f(t+2L) = f(t)$.

Theorem: Any decent periodic function $f(t)$ of period $2\pi$ has can be written in exactly one way as a *Fourier series*:

$$f(t) = \frac{a_0}{2} + a_1 \cos(t) + a_2 \cos(2t) + \ldots$$
$$+ b_1 \sin(t) + b_2 \sin(2t) + \ldots$$

If the need arises, the "Fourier coefficients" can be computed as integrals:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) \, dt , \quad n \geq 0$$
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) \, dt , \quad n > 0$$

[2] Squarewave: A basic example is given by the "standard squarewave," which I denote by $s(t)$: it has period $2\pi$ and

$$s(t) = 1 \text{ for } 0 < t < \pi$$
$$= -1 \text{ for } -\pi < t < 0$$
$$= 0 \text{ for } t = 0, t = \pi$$

This is a standard building block for all sorts of "on/off" periodic signals.

It's odd, so $a_n = \int_{-\pi}^{\pi} f(t) \cos(nt) \, dt$ odd. Even $dt = 0$ for all $n$.

If $f(t)$ is an odd function of period $2\pi$, we can simplify the integral for $b_n$ a little bit. The integrand $f(t) \sin(nt)$ is even, so the integral is twice the integral from 0 to $\pi$:

$$b_n = \frac{2}{\pi} \int_{0}^{\pi} f(t) \sin(nt) \, dt$$

Similarly, if $f(t)$ is even then

$$a_n = \frac{2}{\pi} \int_{0}^{\pi} f(t) \cos(nt) \, dt$$

In our case this is particularly convenient, since $s(t)$ itself needs different definitions depending on the sign of $t$. We have:

$$b_n = \frac{2}{\pi} \int_{0}^{\pi} \sin(nt) \, dt$$
$$= \frac{2}{\pi} \left[ -\frac{\cos(nt)}{n} \right]_0^{\pi}$$
$$= \frac{2}{\pi n} \left[ -\cos(n \pi) - (-1) \right]$$
This depends upon $n$:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\cos(n\pi)$</th>
<th>$1 - \cos(n\pi)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>-1</td>
<td>2</td>
</tr>
</tbody>
</table>

and so on. Thus: $b_n = 0$ for $n$ even and $4\pi/n$ for $n$ odd.

$$\text{sq}(t) = (4/\pi) \left[ \sin(t) + (1/3) \sin(3t) + (1/5) \sin(5t) + \ldots \right]$$

This is the Fourier series for the standard squarewave.

I used the Mathlet FourierCoefficients to illustrate this. Actually, I built up the function

$$(\pi/4) \text{sq}(t) = \sin(t) + (1/3) \sin(3t) + (1/5) \sin(5t) + \ldots \quad (***)$$

and observed the fit.

[3] What is "decent"?

This is quite amazing: the entire function is recovered from a *discrete* sequence of slider settings. They record the strength of the harmonics above the fundamental tone. The sequence of Fourier coefficients is a "transform" of the function, one which only applies (in this form at least) to periodic functions. We'll see another example of a transform later, the Laplace transform.

Let's be more precise about decency. First, a function is *piecewise continuous* if it is broken into continuous segments and such that at each point $t = a$ of discontinuity,

$$f(a-) = \lim_{t \to a \text{ from below}} f(t) \quad \text{and} \quad f(a+) = \lim_{t \to a \text{ from above}} f(t)$$

exist. They exist at points $t = a$ where $f(t)$ is continuous, too, and there they are equal. So $f(t) = 1/t$ is NOT piecewise continuous, but $\text{sq}(t)$ is.

A function is "decent" if it is piecewise continuous and is such that at each point of discontinuity, $t = a$, the value at $a$ is the average of the left and right limits:

$$f(a) = (1/2) (f(a+) + f(a-))$$

So the square wave is decent, and any continuous function is decent.

Addendum to the theorem:
At points of discontinuity, the Fourier series can't make up its mind, so it converges to the average of $f(a^+)$ and $f(a^-)$.

For example, evaluate the Fourier series for $\text{sq}(t)$ at $t = \pi/2$:

\[
\begin{align*}
\sin\left(\frac{\pi}{2}\right) &= +1 \\
\sin\left(\frac{3\pi}{2}\right) &= -1 \\
\sin\left(\frac{5\pi}{2}\right) &= +1 \\
&\ldots
\end{align*}
\]

so

\[
1 = \frac{4}{\pi} \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right) \quad \text{or} \\
1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}
\]

Did you know this? It's due to Newton and Leibnitz.

[4] Tricks: Any way to get an expression (*) will give the same answer!

Example [trig id]: $\cos(t - \pi/4)$.

How to write it like (*)? Well, there's a trig identity we can use:

\[
a \cos(t) + b \sin(t) = A \cos(t - \phi)
\]

if $(a,b)$ has polar coord's $(A,\phi)$

\[
a = A \cos(\phi), \quad b = A \sin(\phi)
\]

For us, $A = 1$, $\phi = \pi/4$, so $a = b = 1/\sqrt{2}$ and

\[
\cos(t - \pi/4) = \left(\frac{1}{\sqrt{2}}\right) \cos(t) + \left(\frac{1}{\sqrt{2}}\right) \sin(t).
\]

That's it: that's the Fourier series. This means $a_1 = b_1 = \sqrt{2}$ and all the others are zero.

Example [linear combinations]:

\[
1 + 2 \text{sq}(t) = 1 + \frac{8}{\pi} \left( \sin(t) + \frac{1}{3} \sin(3t) + \ldots \right)
\]

Example [shifts]: $f(t) = \text{sq}(t + \pi/2)$

\[
= \left(\frac{1}{2}\right) \frac{4}{\pi} \left( \sin(t + \pi/2) + \frac{1}{3} \sin(3(t + \pi/2)) + \ldots \right)
\]

\[
\sin(\theta + \pi/2) = \cos(\theta), \quad \sin(\theta - \pi/2) = -\cos(\theta)
\]

so

\[
f(t) = \frac{4}{\pi} \left( \cos(t) - \frac{1}{3} \cos(3t) + \frac{1}{5} \cos(5t) - \ldots \right)
\]

Example [Stretching]: What about functions of other periods? Suppose $g(x)$ has period $2L$.

Building blocks: $\cos(n(\pi/L)x)$ and $\sin(n(\pi/L)x)$ are periodic of period $2L$. 
Then the Fourier series for $g(x)$ is:

$$g(x) = a_0/2 + a_1 \cos((\pi/L) x) + a_2 \cos((2\pi/L) x) + \ldots$$

$$+ b_2 \sin((\pi/L) x) + b_2 \sin((2\pi/L) x) + \ldots$$

Example: $\text{sq}((\pi/2) x)$ has period 4, not 2\pi: $L = 2$. But we can still write (using the *substitution* $t = (\pi/2) x$):

$$\text{sq}(2\pi x) = (4/\pi) \left( \sin((\pi/2)x) + (1/3) \sin(3(\pi/2) x) + \ldots \right)$$

There are integral formulas as well. [Slide]