Solutions of Spring 2008 Final Exam

1. (a) The isocline for slope 0 is the pair of straight lines \( y = \pm x \). The direction field along these lines is flat.
   The isocline for slope 2 is the hyperbola on the left and right of the straight lines. The direction field along this hyperbola has slope 2.
   The isocline for slope \(-2\) is the hyperbola above and below the straight lines. The direction field along this hyperbola has slope \(-2\).

(b) The sketch should have the following features:
   The curve passes through \((-2,0)\). The slope at \((-2,0)\) is \((-2)^2 - (0)^2 = 4\).
   Going backward from \((-2,0)\), the curve goes down \((dy/dx > 0)\), crosses the left branch of the hyperbola \(x^2 - y^2 = 2\) with slope 2, and gets closer and closer to the line \(y = x\) but never touches it.
   Going forward from \((-2,0)\), the curve first goes up, crosses the left branch of the hyperbola \(x^2 - y^2 = 2\) with slope 2, and becomes flat when it intersects with \(y = -x\). Then the curve goes down and stays between \(y = -x\) and the upper branch of the hyperbola \(x^2 - y^2 = -2\), until it becomes flat as it crosses \(y = x\). Finally, the curve goes up again and stays between \(y = x\) and the right branch of the hyperbola \(x^2 - y^2 = 2\) until it leaves the box.

(c) \(f(100) \approx 100\)

(d) It follows from the picture in (b) that \(f(x)\) reaches a local maximum on the line \(y = -x\). Therefore \(f(a) = -a\).

(e) Since we know \(f(-2) = 0\), to estimate \(f(-1)\) with two steps, the step size is 0.5. At each step, we calculate
\[
x_n = x_{n-1} + 0.5, \quad y_n = y_{n-1} + 0.5(x_{n-1}^2 - y_{n-1}^2)
\]

The calculation is displayed in the following table.

<table>
<thead>
<tr>
<th>(n)</th>
<th>(x_n)</th>
<th>(y_n)</th>
<th>(0.5(x_{n-1}^2 - y_{n-1}^2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(-2)</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>(-1.5)</td>
<td>2</td>
<td>(-0.875)</td>
</tr>
<tr>
<td>2</td>
<td>(-1)</td>
<td>1.125</td>
<td></td>
</tr>
</tbody>
</table>

The estimate of \(f(-1)\) is \(y_2 = 1.125\).

2. (a) The equation is \(\dot{x} = x(x - 1)(x - 2)\). The phase line has three equilibria \(x = 0, 1, 2\).
   For \(x < 0\), the arrow points down.
   For \(0 < x < 1\), the arrow points up.
   For \(1 < x < 2\), the arrow points down.
   For \(x > 2\), the arrow points up.

(b) The horizontal axis is \(t\) and the vertical axis is \(x\). There are three constant solutions \(x(t) \equiv 0, 1, 2\). Their graphs are horizontal.
   Below \(x = 0\), all solutions are decreasing and they tend to \(-\infty\).
Between $x = 0$ and $x = 1$, all solutions are increasing and they approach $x = 1$.
Between $x = 1$ and $x = 2$, all solutions are decreasing and they approach $x = 1$.
Above $x = 2$, all solutions are increasing and they tend to $+\infty$.

(c) A point of inflection $(a, x(a))$ is where $\ddot{x}$ changes sign. In particular, $\ddot{x}(a)$ must be zero. Differentiating the given equation with respect to $t$, we have

$$\ddot{x} = 2\dot{x} - 6x\dot{x} + 3x^2\dot{x} = \dot{x}(2 - 6x + 3x^2)$$

If $x(t)$ is not a constant solution, $\ddot{x}(a) \neq 0$ so that $x(a)$ must satisfy

$$2 - 6x(a) + 3x(a)^2 = 0 \quad \Leftrightarrow \quad x(a) = 1 \pm \frac{1}{\sqrt{3}}.$$  

(d) Typo in the original version: The material being added into the reactor should be Bo instead of Ct.

Let $x(t)$ be the number of moles of Bo in the reactor at time $t$. The rate of loading is 2 moles per year. Hence $x(t)$ satisfies $\dot{x} = -kx + 2$, where $k$ is the decay rate of Bo. Since the half life of Bo is 2 years, $e^{-k\cdot2} = 1/2$ so that $k = (\ln 2)/2$. Therefore we have

$$\dot{x} = -\frac{\ln 2}{2} x + 2.$$  

The initial condition is $x(0) = 0$.

(e) The differential equation is linear. Since we have

$$y' + \left(\frac{3}{x}\right) y = x$$

an integrating factor is given by

$$\exp\left(\int \frac{3}{x} \, dx\right) = \exp(3 \ln x) = x^3.$$  

Multiply the above equation by $x^3$ and integrate:

$$(x^3y)' = x^3y' + 3x^2y = x^4 \quad \Rightarrow \quad x^3y = \frac{1}{5} x^5 + c.$$  

Since $y(1) = 1$, we have $c = 4/5$ and

$$y = \frac{1}{5} x^2 + \frac{4}{5} x^{-3}.$$  

3. (a) Express all complex numbers in polar form:

$$\frac{ie^{2it}}{1 + i} = \frac{e^{i\pi/2} e^{2it}}{\sqrt{2}e^{i\pi/4}} = \frac{1}{\sqrt{2}} e^{i(2t + \pi/2 - \pi/4)} = \frac{1}{\sqrt{2}} e^{i(2t + \pi/4)}$$

The real part is

$$\text{Re} \left(\frac{ie^{2it}}{1 + i}\right) = \frac{1}{\sqrt{2}} \cos \left(2t + \frac{\pi}{4}\right).$$
(b) The trajectory is an outgoing, clockwise spiral that passes through 1.

(c) The polar form of $8i$ is $8e^{i\pi/2}$. Its three cubic roots are

\[
2e^{i\pi/6} = 2 \cos \frac{\pi}{6} + 2i \sin \frac{\pi}{6} = \sqrt{3} + i,
\]
\[
2e^{i(\pi/6 + 2\pi/3)} = 2 \cos \frac{5\pi}{6} + 2i \sin \frac{5\pi}{6} = -\sqrt{3} + i,
\]
\[
2e^{i(\pi/6 + 4\pi/3)} = 2e^{3i\pi/2} = -2i.
\]

4. (a) Let $x_p(t) = at^2 + bt + c$. Plug it into the left hand side of the equation

\[
\ddot{x} + 2\dot{x} + 2x = (2a) + 2(2at + b) + 2(at^2 + bt + c)
\]
\[
= 2at^2 + (4a + 2b)t + (2a + 2b + 2c)
\]

and compare coefficients

\[
2a = 1, \quad 4a + 2b = 0, \quad 2a + 2b + 2c = 1.
\]

The solution is $a = 1/2$, $b = -1$, $c = 1$. Therefore $x_p(t) = \frac{1}{2}t^2 - t + 1$.

(b) The characteristic polynomial is $p(s) = s^2 + 2s + 2$. Using the ERF and linearity,

\[
x_p(t) = \frac{e^{-2t}}{p(-2)} + \frac{1}{p(0)} = \frac{e^{-2t}}{2} + \frac{1}{2}
\]

(c) Consider the complex equation

\[
\ddot{z} + 2\dot{z} + 2z = e^t.
\]

For any solution $z_p$, its imaginary part $x_p = \text{Im } z_p$ satisfies the real equation

\[
\ddot{x} + 2\dot{x} + 2x = \sin t.
\]

The ERF provides a particular solution of the complex equation

\[
z_p(t) = \frac{e^{it}}{p(i)} = \frac{e^{it}}{1 + 2i} = \frac{e^{it}}{\sqrt{5}} e^{i\phi} = \frac{1}{\sqrt{5}} e^{i(t-\phi)}
\]

where $\phi$ is the polar angle of $1 + 2i$. Take the imaginary part of $z_p$

\[
x_p(t) = \text{Im } z_p(t) = \frac{1}{\sqrt{5}} \sin(t - \phi)
\]

This is a sinusoidal solution of the real equation. Its amplitude is $1/\sqrt{5}$.

(d) If $x(t) = t^3$ is a solution, then $q(t) = \ddot{x} + 2\dot{x} + 2x = 6t + 6t^2 + t^3$.

(e) The general solution is $x(t) = t^3 + x_h(t)$, where $x_h(t)$ is a solution of the associated homogeneous equation. Since the characteristic polynomial $s^2 + 2s + 2$ has roots $-1 \pm i$,

\[
x(t) = t^3 + x_h(t) = t^3 + c_1 e^{-t} \cos t + c_2 e^{-t} \sin t.
\]
5. (a) See the formula sheet for the definition of sq(t). The graph of \( f(t) \) is a square wave of period \( 2\pi \). It has a horizontal line segment of height 1 in the range \( -\pi/2 < t < \pi/2 \) and a horizontal line segment of height \(-1\) in the range \( \pi/2 < t < 3\pi/2 \).

(b) Replace \( t \) by \( t + \pi/2 \) in the definition of sq(t)

\[
f(t) = \text{sq}\left(t + \frac{\pi}{2}\right) = \frac{4}{\pi} \left[ \sin\left(t + \frac{\pi}{2}\right) + \frac{1}{3} \sin\left(3t + \frac{3\pi}{2}\right) + \frac{1}{5} \sin\left(5t + \frac{5\pi}{2}\right) + \ldots \right]
\]

\[
= \frac{4}{\pi} \left( \cos t - \frac{1}{3} \cos 3t + \frac{1}{5} \cos 5t + \ldots \right)
\]

(c) First consider the complex equation

\[
\ddot{z} + z = e^{int} \quad \text{for a positive integer } n.
\]

The characteristic polynomial is \( p(s) = s^2 + 1 \). One of the ERGs provides a particular solution of the complex equation

\[
z_p(t) = \frac{e^{int}}{p(in)} = \frac{e^{int}}{1 - n^2}, \quad n \neq 1
\]

\[
z_p(t) = \frac{te^{it}}{p'(i)} = \frac{te^{int}}{2i}, \quad n = 1
\]

The imaginary parts of these functions

\[
u_p(t) = \text{Im} \left( \frac{e^{int}}{1 - n^2} \right) = \frac{\sin nt}{1 - n^2}, \quad n \neq 1
\]

\[
u_p(t) = \text{Im} \left( \frac{te^{it}}{2i} \right) = -\frac{1}{2} t \cos t, \quad n = 1
\]

satisfy the imaginary part of the above complex equation, namely

\[
\ddot{u} + u = \sin nt.
\]

By linearity, a solution of \( \ddot{x} + x = \text{sq}(t) \) is given by

\[
x_p(t) = \frac{4}{\pi} \left( -\frac{1}{2} t \cos t + \frac{1}{3} \frac{\sin 3t}{1 - 3^2} + \frac{1}{5} \frac{\sin 5t}{1 - 5^2} + \ldots \right).
\]

6. (a) For \( t < 0 \), the graph is flat on \( t \)-axis.

For \( 0 < t < 1 \), the graph is flat at 1 unit above \( t \)-axis.

For \( 1 < t < 3 \), the graph is flat at 1 unit below \( t \)-axis.

For \( 3 < t < 4 \), the graph is flat at 1 unit above \( t \)-axis.

For \( t > 4 \), the graph is flat on \( t \)-axis.

(b) \( v(t) = [u(t) - u(t - 1)] - [u(t - 1) - u(t - 3)] + [u(t - 3) - u(t - 4)] \]

\[
= u(t) - 2u(t - 1) + 2u(t - 3) - u(t - 4)
\]
(c) The graph coincides with t-axis for all t, except for two upward spikes at \( t = 0, 3 \) and two downward spikes at \( t = 1, 4 \).

(d) \( \dot{v}(t) = \delta(t) - 2\delta(t - 1) + 2\delta(t - 3) - \delta(t - 4) \)

(e) By the fundamental solution theorem (a.k.a. Green’s formula),

\[
x(t) = (q \ast w)(t) = \int_0^t q(t - \tau)w(\tau) \, d\tau = \int_{a(t)}^{b(t)} w(\tau) \, d\tau.
\]

Now \( q(t - \tau) = 1 \) only for \( 0 < t - \tau < 1 \), or \( t - 1 < \tau < t \), and it is zero elsewhere. Therefore the upper limit \( b(t) \) equals \( t \). The lower limit \( a(t) \) is \( t - 1 \) if \( t - 1 > 0 \), or \( 0 \) if \( t - 1 < 0 \). In other words, \( a(t) = (t - 1)u(t - 1) \).

7. (a) The transfer function is \( W(s) = \frac{1}{p(s)} = \frac{1}{2s^2 + 8s + 16} \).

(b) The unit impulse response \( w(t) \) is the inverse Laplace transform of \( W(s) \). In other words,

\[
\mathcal{L}(w(t)) = \frac{1}{2s^2 + 8s + 16} = \frac{1}{2[(s + 2)^2 + 4]}
\]

\[\Rightarrow \mathcal{L}(e^{2t}w(t)) = \frac{1}{2(s^2 + 4)} = \frac{1}{4} \mathcal{L}(\sin 2t)\]

Therefore \( e^{2t}w(t) = \frac{1}{4} \sin 2t \), and \( w(t) = \frac{1}{4} e^{-2t} \sin 2t \).

(c) Take the Laplace transform of

\[p(D)x = 2\ddot{x}(t) + 8\dot{x}(t) + 16x(t) = \sin t\]

with the initial conditions \( x(0+) = 1 \), \( \dot{x}(0+) = 2 \). This yields

\[2[s^2X(s) - s - 2] + 8[sX(s) - 1] + 16X(s) = \frac{1}{s^2 + 1}\]

\[\Rightarrow \quad X(s) = \frac{1}{2s^2 + 8s + 16} \left( \frac{1}{s^2 + 1} + 2s + 12 \right)\]

8. (a) The characteristic polynomial of \( A \) is

\[\det(A - \lambda I) = \det \begin{bmatrix} 2 - \lambda & 12 \\ 3 & 2 - \lambda \end{bmatrix} = (2 - \lambda)^2 - 36 = (\lambda - 8)(\lambda + 4).\]

Therefore the eigenvalues are \( \lambda = 8, -4 \).

(b) For \( \lambda = 8 \), solve \( (A - 8I)v = 0 \). Since \( A - 8I = \begin{bmatrix} -6 & 12 \\ 3 & -6 \end{bmatrix} \), a solution is \( v = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \).

For \( \lambda = -4 \), solve \( (A + 4I)v = 0 \). Since \( A + 4I = \begin{bmatrix} 6 & 12 \\ 3 & 6 \end{bmatrix} \), a solution is \( v = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \).
(c) The following is a fundamental matrix for $\dot{u} = Bu$

$$F(t) = \begin{bmatrix} e^t & -e^{2t} \\ e^t & e^{2t} \end{bmatrix}$$

Then $e^{tB}$ can be computed as $F(t)F(0)^{-1}$.

$$F(0) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad F(0)^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$e^{tB} = F(t)F(0)^{-1} = \begin{bmatrix} e^t & -e^{2t} \\ e^t & e^{2t} \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^t + e^{2t} & e^t - e^{2t} \\ e^t - e^{2t} & e^t + e^{2t} \end{bmatrix}$$

(d) The general solution of $\dot{u} = Bu$ is

$$u(t) = c_1 \begin{bmatrix} e^t \\ e^t \end{bmatrix} + c_2 \begin{bmatrix} -e^{2t} \\ e^{2t} \end{bmatrix} = F(t) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

The given initial condition implies

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = F(0) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = F(0)^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$$

Therefore the solution of the initial value problem is $u(t) = \frac{1}{2} \begin{bmatrix} 3e^t + e^{2t} \\ 3e^t - e^{2t} \end{bmatrix}$.

9. (a) The phase portrait has the following features:

- All trajectories start at $(0,0)$ and run off to infinity.
- There are straight line trajectories along the lines $y = \pm x$.
- All other trajectories are tangent to $y = x$ at $(0,0)$.
- No two trajectories cross each other.

(b) $Tr A = a + 1, \quad det A = a + 4, \quad \Delta = (Tr A)^2 - 4(det A) = (a - 5)(a + 3)$

(i) $det A < 0 \iff a < -4$

(ii) not for any $a$

(iii) $\Delta > 0, Tr A < 0$ and $det A > 0 \iff -4 < a < -3$

(iv) $\Delta < 0$ and $Tr A < 0 \iff -3 < a < -1$; counterclockwise

(v) $\Delta < 0$ and $Tr A > 0 \iff -1 < a < 5$

(vi) $\Delta = 0$ and $Tr A > 0 \iff a = 5$

10. (a) The equilibria are the solutions of

$$\dot{x} = x^2 - y^2 = 0, \quad \dot{y} = x^2 + y^2 - 8 = 0.$$

This implies $(x^2, y^2) = (4, 4)$, so that $(x, y) = (2, 2), (2, -2), (-2, 2), (-2, -2)$.

(b) The Jacobian is $J(x, y) = \begin{bmatrix} 2x & -2y \\ 2x & 2y \end{bmatrix}$. In particular, $J(-2, -2) = \begin{bmatrix} -4 & 4 \\ -4 & -4 \end{bmatrix}$. 
(c) The linearization of the nonlinear system at \((-2, -2)\) is the linear system \(\dot{u} = J(-2, -2)u\). A computation shows that the eigenvalues of \(J(-2, -2)\) are \(-4 \pm 4i\). The first component of \(u(t)\) is of the form

\[c_1 e^{-4t} \cos 4t + c_2 e^{-4t} \sin 4t = Ae^{-4t} \cos(4t - \phi).\]

This means \(x(t) \approx -2 + Ae^{-4t} \cos(4t - \phi)\) near \((-2, -2)\).

(d) Let \(f(x) = 2x - 3x^2 + x^3\). The phase line in problem 2(a) shows that \(\dot{x} = f(x)\) has a stable equilibrium at \(x = 1\).

The linearization of the nonlinear equation at \(x = 1\) is the linear equation \(\dot{u} = f'(1)u = -u\). Its solutions are \(u(t) = Ae^{-t}\). This means \(x(t) \approx 1 + Ae^{-t}\) near \(x = 1\).