18.03 Problem Set 6: Part II Solutions

Part I points: 22. 8, 23. 0, 24. 8, 25. 0.

22. (a) [4] From II.21(f), \( g(t) = \frac{4}{\pi} (\sin(t) - \frac{1}{3} \sin(3t) + \frac{1}{5} \sin(5t) - \cdots) \). By Superposition III and the fact that \( A \frac{\sin(\omega t)}{\omega^2} \) is a solution to \( \ddot{x} + \omega^2 x = A \sin(\omega t) \), we find that a solution to \( \ddot{x} + \omega_n^2 x = g(t) \) is given by \( x_p = \frac{4}{\pi} (\frac{\sin(t)}{\omega_1^2} - \frac{1}{3} \frac{\sin(3t)}{\omega_3^2} + \cdots) \), as long as \( \omega_n \) is not an odd integer.

(b) [4] If \( \omega_n \) is an odd integer there is no periodic solution.

(c) [4] \( \omega_r = 1 \). For \( \omega \) just less than 1, the term \( \frac{4}{\pi} \frac{\sin(t)}{\omega^2 - 1} \) dominates, and \( x_p \) is relatively close to this: This is antiphase with sin(t) and has large amplitude. When \( \omega_n \) is just greater than 1, the same term occurs and dominates but now is a positive multiple of sin(t), so the system response is in phase with the input.

(d) [4] This is a tricky question. When \( \omega_n \) is not an odd integer, the solution \( x_p \) above is periodic of period 2\( \pi \). The general solution of the homogeneous equation is \( a \cos(\omega_n t) + b \sin(\omega_n t) \), which is periodic of period \( \frac{2\pi}{\omega_n} \). The sum is periodic if some multiple of \( \frac{2\pi}{\omega_n} \) is equal to some multiple of \( \frac{2\pi}{\omega_n} \), and this happens when \( \omega_n \) is a rational number (but not an odd integer).

(e) [4] Yes. [They are periodic of period 2\( \pi \) if \( \omega_n \) is an even integer.]

23. (a) [3] \( f(t) = -u(t)t \), \( f'(t) = -u(t) \).

(b) [3] \( f(t) = u(t)(1-t) \), \( f'(t) = -u(t) + \delta(t) \).

(c) [3] \( f(t) = (u(t) - u(t-1))(2t-1) \), \( f'(t) = 2(u(t) - u(t-1)) - \delta(t) - \delta(t-1) \).

(d) [3] \( f(t) = (u(t) - u(t-1))t + (u(t-1) - u(t-2))(t-1) + (u(t-1) - u(t-2))(t-2) + \cdots = u(t)t - u(t-1) - u(t-2) - \cdots \).

24. (a) [4] The roots of the characteristic polynomial are \( -1 \pm i \), so the general solution to the homogeneous equation is \( e^{-t}(a \cos t + b \sin t) \). The unit impulse response for this second order operator has \( w(0) = 0 \) and \( \dot{w}(0+) = \frac{1}{2} \). The first forces \( a = 0 \) and the second gives \( b = \frac{1}{2} \); \( w(t) = \frac{1}{2} u(t)e^{-t} \sin t \).

(b) [4] For \( t > 0 \), the unit step response is a solution to \( p(D)x = 1 \). In our case, \( x_p = \frac{1}{4} \) is such a solution, and the general solution is then \( x = \frac{1}{4} + e^{-t}(a \cos t + b \sin t) \). We require rest initial conditions: \( 0 = x(0) = \frac{1}{4} + a \) or \( a = -\frac{1}{4} \). \( \dot{x} = e^{-t}((-a+b) \cos t + (-a-b) \sin t) \), so \( 0 = \dot{x}(0) = -a + b \) and \( b = -\frac{1}{4} \) as well: \( v = \frac{1}{4} u(t)(1 - e^{-t}(\cos t + \sin t)) \).

(c) [4] \( \dot{v} = -\frac{1}{4} e^{-t}((1 + 1) \cos t + (1 - 1) \sin t) = \frac{1}{2} e^{-t} \sin t \).

(d) (i) [4] This function has a jump in value, so the operator must be of first order. \( (aD + bI)(2u) = 2a \delta(t) + 2bu(t) \), so \( b = 0 \) and \( a = \frac{1}{2} \); \( p(D) = \frac{1}{2}D \).

(ii) [4] This function has no jump but its derivative does, so the operator must be of second order. For \( t > 0 \), \( w(t) = t \) is the solution to \( a_2 \ddot{x} + a_1 \dot{x} + a_0 x = 0 \) with \( x(0) = 0 \) and \( \dot{x}(0) = \frac{1}{a_2} \). Plugging in: \( a_1 + a_0 t = 0 \) implies \( a_1 = a_0 = 0 \), and \( 1 = \frac{d}{dt} \big|_{t=0} = \frac{1}{a_2} \) implies that \( a_2 = 1 \). So \( p(D) = D^2 \). Or you can argue that \( w(t) = u(t)t \), \( \dot{w}(t) = u(t) \) and \( \ddot{w}(t) = \delta(t) \), so \( a_2 \delta(t) = \delta(t) \) and \( a_2 = 1 \).
(iii) [4] This function $w(t)$ has no jump in value or derivative, but its second derivative
does jump: $\ddot{w}(t) = 2u(t)$. So $w^{(3)}(t) = 2\delta(t)$. This means that we are looking for a third
order operator, $a_3 D^3 + a_2 D^2 + a_1 D + a_0 I$. $t^2$ is a solution to the homogeneous equation,
so $a_2 \cdot 2 + a_1 \cdot 2t + a_0 t^2 = 0$, which implies that $a_0 = a_1 = a_2 = 0$. $\ddot{w}(0) = 2$ implies
that $a_3 = \frac{1}{2}$ and $p(D) = \frac{1}{2} D^3$. Or you can argue that $w(t) = u(t)t^2$, $\dot{w}(t) = u(t)2t$,
$\ddot{w}(t) = 2u(t)$, $w^{(3)}(t) = 2\delta(t)$, so $a_3 w^{(3)}(t) = \delta(t)$ implies that $a_3 = \frac{1}{2}$.

25. (a) [6] $x(t) = w(t) * q(t) = \int_0^t w(t-\tau)q(\tau) \, d\tau = \int_0^t e^{-k(t-\tau)} \cos(\omega \tau) \, d\tau = e^{-kt} \int_0^t \Re(e^{k+i\omega \tau}) \, d\tau =
\frac{1}{k^2 + \omega^2} \Re((k - i\omega)((\cos(\omega t) - e^{-kt}) + i \sin(\omega t))) = \frac{1}{k^2 + \omega^2} (k \cos(\omega t) + \omega \sin(\omega t) - ke^{-kt})$. Then $\dot{x} = \frac{1}{k^2 + \omega^2} (-k \omega \sin(\omega t) + \omega^2 \cos(\omega t) + k^2 e^{-kt})$, and indeed
$\dot{x} + kx = \cos(\omega t)$. Also, $x(0) = 0$: the convolution chose the transient just right.

(b) [6] $x(t) = w(t) * q(t) = \int_0^t w(t-\tau)q(\tau) \, d\tau = \int_0^t \sin(\omega n(t-\tau)) \, d\tau = \frac{1}{\omega n} \sin(\omega n(t-\tau)) \big|_0^t = \frac{1}{\omega n} \sin(\omega n t)$ and $\dot{x} = \cos(\omega n t)$, so it is true that
$\dot{x} + \omega_n^2 x = 1$. Also $x(0) = 0$ and $\dot{x}(0) = 0$: so rest initial conditions. Once again the convolution integral has chosen just the right homogeneous solution to produce rest initial conditions.

(c) [6] $t^2 * t = \int_0^t (t-\tau)^2 \tau \, d\tau = \int_0^t (t^3 - 2t^2 + \tau^3) \, d\tau = \frac{1}{4} t^4 - \frac{2}{3} t^4 + \frac{1}{4} t^4 = \frac{1}{12} t^4$.
$t * t^2 = \int_0^t (t-\tau)(t^2) \, d\tau = \int_0^t (t^3 - t^2) \, d\tau = \frac{1}{3} t^4 - \frac{1}{4} t^4 = \frac{1}{12} t^4$.

(d) [6] $t * t = \int_0^t (t-\tau) \, d\tau = \int_0^t (t^2 - t^2) \, d\tau = \frac{1}{2} t^3 - \frac{1}{3} t^3 = \frac{1}{6} t^3$.

Now $(t * t) * t = \frac{1}{6} \int_0^t (t-\tau)^3 \tau \, d\tau = \frac{1}{6} \int_0^t t^3 - 3t^2 + \tau^3 \, d\tau = \frac{1}{6} (\frac{1}{3} t^3 - \frac{3}{2} t^2 + \frac{1}{6} t^3) = \frac{1}{120} t^5$,
while $t * (t * t) = \frac{1}{6} \int_0^t (t-\tau) \tau^3 \, d\tau = \frac{1}{6} (\frac{1}{4} t^4 - \frac{1}{5} t^5) = \frac{1}{120} t^5$. 