18.03 Problem Set 4: Part II Solutions


13. (a) [4] \( \dot{z} + 2z = e^{(3+4i)t} \) has solution \( z_p = e^{(3+4i)t}/((3 + 4i) + 2) = \frac{5+4i}{25+16} e^{3t}(\cos(4t) + i\sin(4t)) \) so \( x_p = \text{Re}(z_p) = e^{3t}\left(\frac{3}{25} \cos(4t) + \frac{4}{25} \sin(4t)\right) \). The homogeneous equation has general solution \( x_h = ce^{-2t} \), so \( x = e^{3t}\left(\frac{3}{25} \cos(4t) + \frac{4}{25} \sin(4t)\right) + ce^{-2t} \).

(b) [4] \( \ddot{z} + z + 2z = e^{it} \) has solution \( z_p = e^{it}/p(i) \); \( p(i) = -1 + i + 2 = 1 + i \) so \( z_p = \frac{1}{1+i} e^{it} = \frac{1-i}{2}(\cos t + i\sin t) \) and \( x_p = \frac{1}{2} \cos t + \frac{1}{2} \sin t \). We can also write \( p(i) = 1 + i = \sqrt{2} e^{i\pi/4} \), so \( z_p = \frac{1}{\sqrt{2}} e^{-i\pi/4} e^{it} = \frac{\sqrt{2}}{2} e^{i(t-\pi/4)} \) and \( x_p = \text{Re}(z_p) = \frac{\sqrt{2}}{2} \cos(t - \pi/4) \). The characteristic polynomial is \( p(s) = s^2 + s + 2 = \left(s + \frac{1}{2}\right)^2 + \frac{3}{4} \) with roots \(-\frac{1}{2} \pm \frac{\sqrt{7}}{2}\), so the homogeneous equation has general solution \( x_h = e^{-t/2} \left(a \cos\left(\frac{\sqrt{7}}{2} t\right) + b \sin\left(\frac{\sqrt{7}}{2} t\right)\right) \) or \( x_h = Ae^{-t/2} \cos\left(\frac{\sqrt{7}}{2} t - \phi\right) \). The general solution is \( x = x_p + x_h \).

(c) [4] I measure the first six zeros to be 2.36, 4.76, 7.12, 9.48, 11.89, 14.19. The successive differences are 2.40, 2.36, 2.36, 2.41, 2.30, with an average of 2.366. My measured pseudoperiod is twice this, 4.73. The damped circular frequency of this system is \( \omega_d = \frac{\sqrt{7}}{2} \) so the pseudoperiod is \( \frac{2\pi}{\omega_d} = \frac{4\pi}{\sqrt{7}} \approx 4.7496 \). Not bad agreement.

(d) [4] I measure the amplitude as 0.71. It looks like \( x_p(0) \approx 0.50 \). Computed amplitude is \( \frac{\sqrt{7}}{2} \approx 0.707 \). Computed value is \( x_p(0) = \frac{\sqrt{7}}{2} \cos(-\frac{\pi}{4}) = \frac{1}{2} \). Good agreement! \( \dot{x}_p(t) = -\frac{\sqrt{7}}{2} \sin(t - \frac{\pi}{4}) \) so \( \dot{x}_p(0) = -\frac{\sqrt{7}}{2} \sin(-\frac{\pi}{4}) = \frac{1}{2} \).

(e) [5] We need to select \( x_0 \) so that \( x_0(0) = -x_p(0) = -\frac{1}{2} \) and \( \dot{x}_h(0) = -\dot{x}_p(0) = -\frac{1}{2} \). It’s more convenient to use the rectangular expression for \( x_h \) for this. \( x_h(0) = a \) so \( a = -\frac{1}{2} \).
\[
\dot{x}_h = e^{-t/2} \left( -\frac{1}{2}a + \frac{\sqrt{7}}{2} b \right) \cos\left(\frac{\sqrt{7}}{2} t\right) + \left(\cdots\right) \sin\left(\frac{\sqrt{7}}{2} t\right)
\]
so \( \dot{x}_h(0) = -\frac{1}{2}a + \frac{\sqrt{7}}{2} b \). Thus \( b = \frac{2}{\sqrt{7}} \left(\frac{1}{2}a + \frac{\sqrt{7}}{2} b\right) \left(\frac{3}{2} - \frac{1}{2}\right) = -\frac{3}{2}\sqrt{7} \). \( x_h = e^{-t/2} \left( -\frac{1}{2} \cos\left(\frac{\sqrt{7}}{2} t\right) - \frac{3}{2}\sqrt{7} \sin\left(\frac{\sqrt{7}}{2} t\right)\right) \).

14. (a) [8] Since the input signal has amplitude 1, the gain is the amplitude of the system response. The equation is \( \ddot{z} + \frac{1}{2} \dot{z} + 4z = 4 \cos(2t) \). The complex replacement is \( \ddot{z} + \frac{1}{2} \dot{z} + 4z = 4e^{2it} \). Since \( p(2i) = (2i)^2 + \frac{1}{2}(2i) + 4 = i, z_p = 4e^{2it}/i \), so \( x_p = \text{Re}(z_p) = 4\sin(2t) \) the gain is 4. Also the phase lag of the sine behind the cosine is \( \phi = \frac{\pi}{2} \). The time lag is \( t_0 = \frac{\omega}{\omega} = \frac{\pi}{4} \approx 0.7854 \).

(b) [8] \( H(\omega) = \frac{4}{p(i\omega)} = \frac{4}{4 - \omega^2 + \frac{1}{2} i\omega} \). The gain is \( |H(\omega)| = \frac{4}{\sqrt{(4 - \omega^2)^2 + \frac{1}{4}\omega^2}} \).

The phase lag \( \phi = -\text{Arg}(H(\omega)) = \text{Arg}(p(i\omega)) \) so \( \tan \phi = \frac{\text{Im}(p(i\omega))}{\text{Re}(p(i\omega))} = \frac{\omega/2}{4 - \omega^2} \).

15. (a) [6] \( p(s) = s + 1 \) and \( p(-1) = 0 \), so we are in resonance. \( p'(s) = 1 \) so the ERF/Resonant gives \( x_p = te^{-t} \).

(b) [6] We can’t apply undetermined coefficients directly since \( p(0) = 0 \). Let \( u = \dot{x} \), so \( \ddot{u} - u = t^2 + 1 \). Try \( u = at^2 + bt + c \), so \( \ddot{u} = 2a \) and \( t^2 + 1 = \ddot{u} - u = -at^2 - bt + (2a - c) \) implies \( a = -1, b = 0, 2a - c = 1 \) or \( c = -3 \): so \( u_p = -t^2 - 3t \). Then \( x_p \) is the integral of \( u_p \); \( x_p = \frac{-1}{3}t^3 - 3t \). To solve the homogeneous equation, factor \( p(s) = s(s-1)(s+1) \) so \( x_h = c_1 + c_2 e^{t} + c_3 e^{-t} \). \( x = x_p + x_h \).