18.01 Single Variable Calculus
Fall 2006

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Lecture 34: Indeterminate Forms - L’Hôpital’s Rule

L’Hôpital’s Rule

(Two correct spellings: “L’Hôpital” and “L’Hospital”)

Sometimes, we run into indeterminate forms. These are things like

\[
\frac{0}{0}
\]

and

\[
\frac{\infty}{\infty}
\]

For instance, how do you deal with the following?

\[
\lim_{x \to 1} \frac{x^3 - 1}{x^2 - 1} = \frac{0}{0}?
\]

Example 0. One way of dealing with this is to use algebra to simplify things:

\[
\lim_{x \to 1} \frac{x^3 - 1}{x^2 - 1} = \lim_{x \to 1} \frac{(x - 1)(x^2 + x + 1)}{(x - 1)(x + 1)} = \lim_{x \to 1} \frac{x^2 + x + 1}{x + 1} = \frac{3}{2}
\]

In general, when \(f(a) = g(a) = 0\),

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}
\]

This is the easy version of L’Hôpital’s rule:

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}
\]

Note: this only works when \(g'(a) \neq 0\)!

In example 0,

\[
f(x) = x^3 = 1; \quad g(x) = x^2 - 1\]

\[
f'(x) = 3x^2; \quad g'(x) = 2x \quad \Rightarrow \quad f'(1) = 3; \quad g'(1) = 2
\]

The limit is \(f'(1)/g'(1) = \frac{3}{2}\). Now, let’s go on to the full L’Hôpital rule.
Example 1. Apply L'Hôpital’s rule (a.k.a. “L’Hop”) to
\[ \lim_{x \to 1} \frac{x^{15} - 1}{x^3 - 1} \]
to get
\[ \lim_{x \to 1} \frac{x^{15} - 1}{x^3 - 1} = \lim_{x \to 1} \frac{15x^{14}}{3x^2} = \frac{15}{3} = 5 \]
Let’s compare this with the answer we’d get if we used linear approximation techniques, instead of L'Hôpital’s rule:
\[ x^{15} - 1 \approx 15(x - 1) \]
(Here, \( f(x) = x^{15} - 1, a = 1, f(a) = b = 0, m = f'(1) = 15, \) and \( f(x) \approx m(x - a) + b. \))
Similarly,
\[ x^3 - 1 \approx 3(x - 1) \]
Therefore,
\[ \frac{x^{15} - 1}{x^3 - 1} \approx \frac{15(x - 1)}{3(x - 1)} = 5 \]

Example 2. Apply L’Hop to
\[ \lim_{x \to 0} \frac{\sin 3x}{x} \]
to get
\[ \lim_{x \to 0} \frac{3 \cos 3x}{1} = 3 \]
This is the same as
\[ \left. \frac{d}{dx} \sin(3x) \right|_{x=0} = 3 \cos(3x) \left|_{x=0} = 3 \right. \]

Example 3.
\[ \lim_{x \to \pi/4} \frac{\sin x - \cos x}{x - \pi/4} = \lim_{x \to \pi/4} \frac{\cos x + \sin x}{1} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2} \]
\[ f(x) = \sin x - \cos x, \quad f'(x) = \cos x + \sin x \]
\[ f'\left(\frac{\pi}{4}\right) = \sqrt{2} \]

Remark: Derivatives \( \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \) are always a \( \frac{0}{0} \) type of limit.

Example 4. \( \lim_{x \to 0} \frac{\cos x - 1}{x} \).
Use L'Hôpital’s rule to evaluate the limit:
\[ \lim_{x \to 0} \frac{\cos x - 1}{x} = \lim_{x \to 0} \frac{-\sin x}{x} = 0 \]
Example 5. \(\lim_{x \to 0} \frac{\cos x - 1}{x^2}\).

\[
\lim_{x \to 0} \frac{\cos x - 1}{x^2} = \lim_{x \to 0} \frac{\cos x - 1}{x^2} = \lim_{x \to 0} -\frac{\sin x}{2x} = \lim_{x \to 0} -\frac{-\cos x}{2} = -\frac{1}{2}
\]

Just to check, let’s compare that answer to the one we would get if we used quadratic approximation techniques. Remember that:

\[
\cos x \approx 1 - \frac{1}{2} x^2 \quad (x \approx 0)
\]

\[
\frac{\cos x - 1}{x^2} \approx \frac{1 - \frac{1}{2} x^2 - 1}{x^2} = \frac{-\frac{1}{2} x^2}{x^2} = -\frac{1}{2}
\]

Example 6. \(\lim_{x \to 0} \frac{\sin x}{x^2}\).

\[
\lim_{x \to 0} \frac{\sin x}{x^2} = \lim_{x \to 0} \frac{\cos x}{2x} \quad \text{By L'Hôpital’s rule}
\]

If we apply L'Hôpital again, we get

\[
\lim_{x \to 0} -\frac{\sin x}{2} = 0
\]

But this doesn’t agree with what we get from taking the linear approximation:

\[
\frac{\sin x}{x^2} \approx \frac{x}{x^2} = \frac{1}{x} \to \infty \quad \text{as} \quad x \to 0^+
\]

We can clear up this seeming paradox by noting that

\[
\lim_{x \to 0} \frac{\cos x}{2x} = \frac{1}{0}
\]

The limit is not of the form \(\frac{0}{0}\), which means L'Hôpital’s rule cannot be used. The point is: look before you L'Hôp!

More “interesting” cases that work.

It is also okay to use L'Hôpital’s rule on limits of the form \(\frac{\infty}{\infty}\), or if \(x \to \infty\), or \(x \to -\infty\). Let’s apply this to rates of growth. Which function goes to \(\infty\) faster: \(x\), \(e^{ax}\), or \(\ln x\)?

Example 7. For \(a > 0\),

\[
\lim_{x \to \infty} \frac{e^{ax}}{x} = \lim_{x \to \infty} \frac{ae^{ax}}{1} = +\infty
\]

So \(e^{ax}\) grows faster than \(x\) (for \(a > 0\)).

Example 8.

\[
\lim_{x \to -\infty} \frac{e^{ax}}{x^{10}} = \text{by L'Hôpital} = \lim_{x \to -\infty} \frac{ae^{ax}}{10x^9} = \lim_{x \to -\infty} \frac{e^{2ax}}{10 \cdot 9x^8} = \cdots = \lim_{x \to -\infty} \frac{a^{10}e^{ax}}{10!} = \infty
\]
You can apply L'Hôpital's rule ten times. There's a better way, though:

\[
\left( \frac{e^{ax}}{x^{10}} \right)^{1/10} = \frac{e^{ax/10}}{x}
\]

\[
\lim_{x \to \infty} e^{ax} = \lim_{x \to \infty} \left( \frac{e^{ax/10}}{x} \right)^{10} = \infty^{10} = \infty
\]

**Example 9.**

\[
\lim_{x \to \infty} \frac{\ln x}{x^{1/3}} = \lim_{x \to \infty} \frac{1/x}{1/3x^{2/3}} = \lim_{x \to \infty} 3x^{-1/3} = 0
\]

Combining the preceding examples, \( \ln x \ll x^{1/3} \ll x \ll e^{ax} \quad (x \to \infty, a > 0) \)

L'Hôpital’s rule applies to \( \frac{0}{0} \) and \( \frac{\infty}{\infty} \). But, we sometimes face other indeterminate limits, such as \( 1^\infty \), \( 0^0 \), and \( 0 \cdot \infty \). Use algebra, exponentials, and logarithms to put these in L'Hôpital form.

**Example 10.** \( \lim_{x \to 0} x^x \) for \( x > 0 \).
Because the exponent is a variable, use base \( e \):

\[
\lim_{x \to 0} x^x = \lim_{x \to 0} e^{x \ln x}
\]

First, we need to evaluate the limit of the exponent

\[
\lim_{x \to 0} x \ln x
\]

This limit has the form \( 0 \cdot \infty \). We want to put it in the form \( \frac{0}{0} \) or \( \frac{\infty}{\infty} \).

Let’s try to put it into the \( \frac{0}{0} \) form:

\[
\frac{x}{1/\ln x}
\]

We don’t know how to find \( \lim_{x \to 0} \frac{1}{\ln x} \), though, so that approach isn’t helpful.

Instead, let’s try to put it into the \( \frac{\infty}{\infty} \) form:

\[
\frac{\ln x}{1/x}
\]

Using L'Hôpital's rule, we find

\[
\lim_{x \to 0} x \ln x = \lim_{x \to 0} \frac{\ln x}{1/x} = \lim_{x \to 0} \frac{1/x}{-1/x^2} = \lim_{x \to 0} (-x) = 0
\]

Therefore,

\[
\lim_{x \to 0} x^x = \lim_{x \to 0} e^{x \ln x} = e^{\lim_{x \to 0} (x \ln x)} = e^0 = 1
\]