Lecture 32: Polar Co-ordinates, Area in Polar Co-ordinates

Polar Coordinates

In polar coordinates, we specify an object’s position in terms of its distance $r$ from the origin and the angle $\theta$ that the ray from the origin to the point makes with respect to the $x$-axis.

**Example 1.** What are the polar coordinates for the point specified by $(1, -1)$ in rectangular coordinates?

![Figure 2: Rectangular Co-ordinates to Polar Co-ordinates.](image)

$$r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$\theta = -\frac{\pi}{4}$$

In most cases, we use the convention that $r \geq 0$ and $0 \leq \theta \leq 2\pi$. But another common convention is to say $r \geq 0$ and $-\pi \leq \theta \leq \pi$. All values of $\theta$ and even negative values of $r$ can be used.
Regardless of whether we allow positive or negative values of $r$ or $\theta$, what is always true is:

$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta$$

For instance, $x = 1, y = -1$ can be represented by $r = -\sqrt{2}, \theta = \frac{3\pi}{4}$:

$$1 = x = -\sqrt{2} \cos \frac{3\pi}{4} \quad \text{and} \quad -1 = y = -\sqrt{2} \sin \frac{3\pi}{4}$$

**Example 2.** Consider a circle of radius $a$ with its center at $x = a, y = 0$. We want to find an equation that relates $r$ to $\theta$. 

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Figure 3: Rectangular Co-ordinates to Polar Co-ordinates.

Figure 4: Circle of radius $a$ with center at $x = a, y = 0$. 

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We know the equation for the circle in rectangular coordinates is

\[(x - a)^2 + y^2 = a^2\]

Start by plugging in:

\[x = r \cos \theta \quad \text{and} \quad y = r \sin \theta\]

This gives us

\[(r \cos \theta - a)^2 + (r \sin \theta)^2 = a^2\]

\[r^2 \cos^2 \theta - 2ar \cos \theta + a^2 + r^2 \sin^2 \theta = a^2\]

\[r^2 - 2ar \cos \theta = 0\]

\[r = 2a \cos \theta\]

The range of \(0 \leq \theta \leq \frac{\pi}{2}\) traces out the top half of the circle, while \(-\frac{\pi}{2} \leq \theta \leq 0\) traces out the bottom half. Let’s graph this.

![Graph of a circle](image)

**Figure 5:** \(r = 2a \cos \theta, \quad -\pi/2 \leq \theta \leq \pi/2.\)

At \(\theta = 0\), \(r = 2a \Rightarrow x = 2a, \ y = 0\)

At \(\theta = \frac{\pi}{4}\), \(r = 2a \cos \frac{\pi}{4} = a\sqrt{2}\)

The main issue is finding the range of \(\theta\) tracing the circle once. In this case, \(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\).

\[\theta = -\frac{\pi}{2} \quad \text{(down)}\]

\[\theta = \frac{\pi}{2} \quad \text{(up)}\]

Weird range (avoid this one): \(\frac{\pi}{2} < \theta < \frac{3\pi}{2}\). When \(\theta = \pi\), \(r = 2a \cos \pi = 2a(-1) = -2a\). The radius points “backwards”. In the range \(\frac{\pi}{2} < \theta < \frac{3\pi}{2}\), the same circle is traced out a second time.
Area in Polar Coordinates

Since radius is a function of angle \( r = f(\theta) \), we will integrate with respect to \( \theta \). The question is: what, exactly, should we integrate?

\[
\int_{\theta_1}^{\theta_2} r^2 \, d\theta
\]

Let’s look at a very small slice of this region:

This infinitesimal slice is approximately a right triangle. To find its area, we take:

\[
\text{Area of slice} \approx \frac{1}{2} (\text{base})(\text{height}) = \frac{1}{2} r(r \, d\theta)
\]

So,

\[
\text{Total Area} = \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 \, d\theta
\]
Example 3. \( r = 2a \cos \theta \), and \(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\) (the circle in Figure 5).

\[
A = \text{area} = \int_{-\pi/2}^{\pi/2} \frac{1}{2} (2a \cos \theta)^2 \, d\theta = 2a^2 \int_{-\pi/2}^{\pi/2} \cos^2 \theta \, d\theta
\]

Because \( \cos^2 \theta = \frac{1}{2} + \frac{1}{2} \cos 2\theta \), we can rewrite this as

\[
A = \text{area} = \int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta) \, d\theta = a^2 \int_{-\pi/2}^{\pi/2} d\theta + a^2 \int_{-\pi/2}^{\pi/2} \cos 2\theta \, d\theta
\]

\[
= \pi a^2 + \frac{1}{2} \sin 2\theta \bigg|_{-\pi/2}^{\pi/2} = \pi a^2 + \frac{1}{2} \left[ \sin \pi - \sin (-\pi) \right]
\]

\[
A = \text{area} = \pi a^2
\]

Example 4: Circle centered at the Origin.

![Circle centered at the Origin](image)

Figure 8: Example 4: Circle centered at the origin

\[
x = r \cos \theta; \quad y = r \sin \theta
\]

\[
x^2 + y^2 = r^2 \cos^2 \theta + r^2 \sin^2 \theta = r^2
\]

The circle is \( x^2 + y^2 = a^2 \), so \( r = a \) and

\[
x = a \cos \theta; \quad y = a \sin \theta
\]

\[
A = \int_{0}^{2\pi} \frac{1}{2} a^2 \, d\theta = \frac{1}{2} a^2 \cdot 2\pi = \pi a^2.
\]
Example 5: A Ray. In this case, $\theta = b$.

![Image of a ray with $\theta = b$](image)

The range of $r$ is $0 \leq r < \infty$; $x = r \cos b$; $y = r \sin b$.

Example 6: Finding the Polar Formula, based on the Cartesian Formula

![Image of Cartesian to Polar conversion](image)

Consider, in cartesian coordinates, the line $y = 1$. To find the polar coordinate equation, plug in $y = r \sin \theta$ and $x = r \cos \theta$ and solve for $r$.

$$r \sin \theta = 1 \implies r = \frac{1}{\sin \theta} \quad \text{with} \quad 0 < \theta < \pi$$
Example 7: Going back to \((x, y)\) coordinates from \(r = f(\theta)\).

Start with

\[
r = \frac{1}{1 + \frac{1}{2} \sin \theta}\]

Hence,

\[
r + \frac{r}{2} \sin \theta = 1
\]

Plug in \(r = \sqrt{x^2 + y^2}\):

\[
\sqrt{x^2 + y^2} + \frac{y}{2} = 1
\]

\[
\sqrt{x^2 + y^2} = 1 - \frac{y}{2} \implies x^2 + y^2 = \left(1 - \frac{y}{2}\right)^2 = 1 - y + \frac{y^2}{4}
\]

Finally,

\[
x^2 + \frac{3y^2}{4} + y = 1
\]

This is an equation for an ellipse, with the origin at one focus.

Useful conversion formulas:

\[
r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)
\]

Example 8: A Rose \(r = \cos(2\theta)\)

The graph looks a bit like a flower:

\[\text{Figure 11: Example 8: Rose}\]

For the first “petal”

\[-\frac{\pi}{4} < \theta < \frac{\pi}{4}\]

Note: Next lecture is Lecture 34 as Lecture 33 is Exam 4.