MIT OpenCourseWare
http://ocw.mit.edu

18.01 Single Variable Calculus
Fall 2006

For information about citing these materials or our Terms of Use, visit: http://ocw.mit.edu/terms.
Lecture 28: Integration by Inverse Substitution; Completing the Square

Trigonometric Substitutions, continued

Figure 1: Find area of shaded portion of semicircle.

\[ \int_{0}^{x} \sqrt{a^2 - t^2} \, dt \]

\[ t = a \sin u; \quad dt = a \cos u \, du \]

\[ a^2 - t^2 = a^2 - a^2 \sin^2 u = a^2 \cos^2 u \implies \sqrt{a^2 - t^2} = a \cos u \quad \text{(No more square root!)} \]

Start: \( x = -a \Leftrightarrow u = -\pi/2; \quad \text{Finish: } \ x = a \Leftrightarrow u = \pi/2 \)

\[ \int \sqrt{a^2 - t^2} \, dt = \int a^2 \cos^2 u \, du = a^2 \int \frac{1 + \cos(2u)}{2} \, du = a^2 \left[ \frac{u}{2} + \frac{\sin(2u)}{4} \right] + c \]

(Recall, \( \cos^2 u = \frac{1 + \cos(2u)}{2} \)).

We want to express this in terms of \( x \), not \( u \). When \( t = 0 \), \( a \sin u = 0 \), and therefore \( u = 0 \). When \( t = x \), \( a \sin u = x \), and therefore \( u = \sin^{-1}(x/a) \).

\[ \frac{\sin(2u)}{4} = \frac{2 \sin u \cos u}{4} = \frac{1}{2} \sin u \cos u \]

\[ \sin u = \sin \left( \sin^{-1}(x/a) \right) = \frac{x}{a} \]
How can we find $\cos u = \cos \left( \sin^{-1}\left(\frac{x}{a}\right) \right)$? Answer: use a right triangle (Figure 2).

![Figure 2: $\sin u = \frac{x}{a}; \cos u = \frac{\sqrt{a^2 - x^2}}{a}$.](image)

From the diagram, we see

$$\cos u = \frac{\sqrt{a^2 - x^2}}{a}$$

And finally,

$$\int_0^x \sqrt{a^2 - t^2} \, dt = a^2 \left[ \frac{u}{4} + \frac{1}{2} \sin u \cos u \right] - 0 = a^2 \left[ \frac{\sin^{-1}\left(\frac{x}{a}\right)}{2} + \frac{1}{2} \frac{x}{a} \frac{\sqrt{a^2 - x^2}}{a} \right]$$

$$\int_0^x \sqrt{a^2 - t^2} \, dt = \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2} x \sqrt{a^2 - x^2}$$

When the answer is this complicated, the route to getting there has to be rather complicated. There’s no way to avoid the complexity.

Let’s double-check this answer. The area of the upper shaded sector in Figure 3 is $\frac{1}{2} a^2 u$. The area of the lower shaded region, which is a triangle of height $\sqrt{a^2 - x^2}$ and base $x$, is $\frac{1}{2} x \sqrt{a^2 - x^2}$. 


Here is a list of integrals that can be computed using a trig substitution and a trig identity.

\[
\begin{array}{ccc}
\text{integral} & \text{substitution} & \text{trig identity} \\
\int \frac{dx}{\sqrt{x^2 + 1}} & x = \tan u & \tan^2 u + 1 = \sec^2 u \\
\int \frac{dx}{\sqrt{x^2 - 1}} & x = \sec u & \sec^2 u - 1 = \tan^2 u \\
\int \frac{dx}{\sqrt{1 - x^2}} & x = \sin u & 1 - \sin^2 u = \cos^2 u \\
\end{array}
\]

Let’s extend this further. How can we evaluate an integral like this?

\[
\int \frac{dx}{\sqrt{x^2 + 4x}}
\]

When you have a linear and a quadratic term under the square root, complete the square.

\[
x^2 + 4x = (\text{something})^2 \pm \text{constant}
\]

In this case,

\[
(x + 2)^2 = x^2 + 4x + 4 \implies x^2 + 4x = (x + 2)^2 - 4
\]

Now, we make a substitution.

\[
v = x + 2 \quad \text{and} \quad dv = dx
\]

Plugging these in gives us

\[
\int \frac{dx}{\sqrt{(x + 2)^2 - 4}} = \int \frac{dv}{\sqrt{v^2 - 4}}
\]

Now, let

\[
v = 2 \sec u \quad \text{and} \quad dv = 2 \sec u \tan u \quad du
\]

\[
\int \frac{dv}{\sqrt{v^2 - 4}} = \int \frac{2 \sec u \tan u \ du}{2 \tan u} = \int \sec u \ du
\]
Remember that
\[ \int \sec u \, du = \ln(\sec u + \tan u) + c \]

Finally, rewrite everything in terms of \( x \).

\[ v = 2 \sec u \Leftrightarrow \cos u = \frac{2}{v} \]

Set up a right triangle as in Figure 4. Express \( \tan u \) in terms of \( v \).

![Figure 4: sec \( u \) = \( v/2 \) or \( \cos u = 2/v \).]

Just from looking at the triangle, we can read off

\[ \sec u = \frac{v}{2} \quad \text{and} \quad \tan u = \frac{\sqrt{v^2 - 4}}{2} \]

\[ \int 2 \sec u \, du = \ln \left( \frac{v}{2} + \frac{\sqrt{v^2 - 4}}{2} \right) + c \]

\[ = \ln(v + \sqrt{v^2 - 4}) - \ln 2 + c \]

We can combine those last two terms into another constant, \( \hat{c} \).

\[ \int \frac{dx}{\sqrt{x^2 + 4x}} = \ln(x + 2 + \sqrt{x^2 + 4x}) + \hat{c} \]

Here’s a teaser for next time. In the next lecture, we’ll integrate all rational functions. By “rational functions,” we mean functions that are the ratios of polynomials:

\[ \frac{P(x)}{Q(x)} \]

It’s easy to evaluate an expression like this:

\[ \int \left( \frac{1}{x - 1} + \frac{3}{x + 2} \right) \, dx = \ln |x - 1| + 3 \ln |x + 2| + c \]
If we write it a bit differently, however, it becomes much harder to integrate:

\[
\frac{1}{x-1} + \frac{3}{x+2} = \frac{x+2+3(x-1)}{(x-1)(x+2)} = \frac{4x-1}{x^2+x-2}
\]

\[
\int \frac{4x-1}{x^2+x-2} = ???
\]

How can we reorganize what to do starting from \((4x-1)/(x^2+x-2)\)? Next time, we'll see how. It involves some algebra.