Lecture 23: Work, Average Value, Probability

Application of Integration to Average Value

You already know how to take the average of a set of discrete numbers:

\[
\frac{a_1 + a_2}{2} \text{ or } \frac{a_1 + a_2 + a_3}{3}
\]

Now, we want to find the average of a continuum.

![Discrete approximation to \( y = f(x) \) on \( a \leq x \leq b \).](image)

Figure 1: Discrete approximation to \( y = f(x) \) on \( a \leq x \leq b \).

\[
\text{Average} \approx \frac{y_1 + y_2 + \ldots + y_n}{n}
\]

where

\[
a = x_0 < x_1 < \cdots x_n = b
\]

\[
y_0 = f(x_0), \ y_1 = f(x_1), \ldots y_n = f(x_n)
\]

and

\[
n(\Delta x) = b - a \quad \iff \quad \Delta x = \frac{b - a}{n}
\]

The limit of the Riemann Sums is

\[
\lim_{n \to \infty} \left( y_1 + \cdots + y_n \right) \frac{b - a}{n} = \int_a^b f(x) \, dx
\]

Divide by \( b - a \) to get the continuous average

\[
\lim_{n \to \infty} \frac{y_1 + \cdots + y_n}{n} = \frac{1}{b-a} \int_a^b f(x) \, dx
\]
Example 1. Find the average of \( y = \sqrt{1 - x^2} \) on the interval \(-1 \leq x \leq 1\). (See Figure 2)

\[
\text{Average height} = \frac{1}{2} \int_{-1}^{1} \sqrt{1 - x^2} \, dx = \frac{1}{2} \left( \frac{\pi}{2} \right) = \frac{\pi}{4}
\]

Example 2. The average of a constant is the same constant

\[
\frac{1}{b - a} \int_{a}^{b} 53 \, dx = 53
\]

Example 3. Find the average height \( y \) on a semicircle, with respect to arclength. (Use \( d\theta \) not \( dx \). See Figure 3)

Figure 2: Average height of the semicircle.

Figure 3: Different weighted averages.
Example 4. Find the average temperature of water in the witches cauldron from last lecture. (See Figure 4).

First, recall how to find the volume of the solid of revolution by disks.

\[ V = \int_{0}^{1} (\pi x^2) \, dy = \int_{0}^{1} \pi y \, dy = \frac{\pi y^2}{2} \bigg|_{0}^{1} = \frac{\pi}{2} \]

Recall that \( T(y) = 100 - 30y \) and \( T(0) = 100^\circ; T(1) = 70^\circ \). The average temperature per unit volume is computed by giving an importance or “weighting” \( w(y) = \pi y \) to the disk at height \( y \).

\[ \frac{\int_{0}^{1} T(y)w(y) \, dy}{\int_{0}^{1} w(y) \, dy} \]

The numerator is

\[ \int_{0}^{1} T \pi y \, dy = \pi \int_{0}^{1} (100 - 30y) \, dy = \pi (500y^2 - 10y^3) \bigg|_{0}^{1} = 40\pi \]

Thus the average temperature is:

\[ \frac{40\pi}{\pi/2} = 80^\circ C \]

Compare this with the average taken with respect to height \( y \):

\[ \frac{1}{1} \int_{0}^{1} T \, dy = \int_{0}^{1} (100 - 30y) \, dy = (100y - 15y^2) \bigg|_{0}^{1} = 85^\circ C \]

\( T \) is linear. Largest \( T = 100^\circ C \), smallest \( T = 70^\circ C \), and the average of the two is

\[ \frac{70 + 100}{2} = 85 \]
The answer $85^\circ$ is consistent with the ordinary average. The weighted average (integration with respect to $\pi y \, dy$) is lower ($80^\circ$) because there is more water at cooler temperatures in the upper parts of the cauldron.

**Dart board, revisited**

Last time, we said that the accuracy of your aim at a dart board follows a “normal distribution”:

$$ce^{-r^2}$$

Now, let’s pretend someone – say, your little brother – foolishly decides to stand close to the dart board. What is the chance that he’ll get hit by a stray dart?

![Diagram of dart board and brother](image)

**Figure 5**: Shaded section is $2r_1 < r < 3r_1$ between 3 and 5 o’clock.

To make our calculations easier, let’s approximate your brother as a sector (the shaded region in Fig. 5). Your brother doesn’t quite stand in front of the dart board. Let us say he stands at a distance $r$ from the center where $2r_1 < r < 3r_1$ and $r_1$ is the radius of the dart board. Note that your brother doesn’t surround the dart board. Let us say he covers the region between 3 o’clock and 5 o’clock, or $\frac{1}{6}$ of a ring.

Remember that

$$\text{probability} = \frac{\text{part}}{\text{whole}}$$
The ring has weight \((ce^{-r^2})(2\pi r)(dr)\) (see Figure 6). The probability of a dart hitting your brother is:

\[
\frac{\int_{2r_1}^{3r_1} ce^{-r^2} 2\pi r \, dr}{\int_{0}^{\infty} ce^{-r^2} 2\pi r \, dr}
\]

Recall that \(\frac{1}{6} = \frac{5 - 3}{12}\) is our approximation to the portion of the circumference where the little brother stands. (Note: \(e^{-r^2} = e^{(-r^2)}\) not \((e^{-r})^2\))

\[
\int_{a}^{b} e^{-r^2} \, dr = -\frac{1}{2} e^{-b^2} + \frac{1}{2} e^{-a^2} \quad \left( \frac{d}{dr} e^{-r^2} = -2re^{-r^2} \right)
\]

Denominator:

\[
\int_{0}^{\infty} e^{-r^2} r \, dr = -\frac{1}{2} e^{-R^2} \Big|_{0}^{R=\infty} = -\frac{1}{2} e^{-R^2} + \frac{1}{2} e^{-0^2} = \frac{1}{2}
\]

(Note that \(e^{-R^2} \to 0\) as \(R \to \infty\).)

\[
\text{Probability} = \frac{\int_{2r_1}^{3r_1} ce^{-r^2} 2\pi r \, dr}{\int_{0}^{\infty} ce^{-r^2} 2\pi r \, dr} = \frac{1}{3} \int_{2r_1}^{3r_1} e^{-r^2} r \, dr = \frac{1}{3} \int_{2r_1}^{3r_1} e^{-r^2} \, dr = \frac{-e^{-r^2}}{6} \bigg|_{2r_1}^{3r_1}
\]
Probability = \frac{-e^{-9r_1^2} + e^{-4r_1^2}}{6}

Let’s assume that the person throwing the darts hits the dartboard \(0 \leq r \leq r_1\) about half the time. (Based on personal experience with 7-year-olds, this is realistic.)

\[
P(0 \leq r \leq r_1) = \frac{1}{2} = \int_0^{r_1} 2e^{-r^2} r \, dr = -e^{-r_1^2} + 1 \implies e^{-r_1^2} = \frac{1}{2}
\]

\[
e^{-r_1^2} = \frac{1}{2}
\]

\[
e^{-9r_1^2} = \left(e^{-r_1^2}\right)^9 = \left(\frac{1}{2}\right)^9 \approx 0
\]

\[
e^{-4r_1^2} = \left(e^{-r_1^2}\right)^4 = \left(\frac{1}{2}\right)^4 = \frac{1}{16}
\]

So, the probability that a stray dart will strike your little brother is

\[
\left(\frac{1}{16}\right) \left(\frac{1}{6}\right) \approx \frac{1}{100}
\]

In other words, there’s about a 1% chance he’ll get hit with each dart thrown.
Volume by Slices: An Important Example

Compute \( Q = \int_{-\infty}^{\infty} e^{-x^2} \, dx \)

![Figure 8: Q = Area under curve \( e^{(-x^2)} \).](image)

This is one of the most important integrals in all of calculus. It is especially important in probability and statistics. It’s an improper integral, but don’t let those \( \infty \)'s scare you. In this integral, they’re actually easier to work with than finite numbers would be.

To find \( Q \), we will first find a volume of revolution, namely,

\[
V = \text{volume under } e^{-r^2} \quad (r = \sqrt{x^2 + y^2})
\]

We find this volume by the method of shells, which leads to the same integral as in the last problem. The shell or cylinder under \( e^{-r^2} \) at radius \( r \) has circumference \( 2\pi r \), thickness \( dr \); (see Figure 9). Therefore \( dV = e^{-r^2} 2\pi r \, dr \). In the range \( 0 \leq r \leq R \),

\[
\int_0^R e^{-r^2} 2\pi r \, dr = -\pi e^{-r^2} \bigg|_0^R = -\pi e^{-R^2} + \pi
\]

When \( R \to \infty \), \( e^{-R^2} \to 0 \),

\[
V = \int_0^\infty e^{-r^2} 2\pi r \, dr = \pi \quad \text{(same as in the darts problem)}
\]
Next, we will find $V$ by a second method, the method of slices. Slice the solid along a plane where $y$ is fixed. (See Figure 10). Call $A(y)$ the cross-sectional area. Since the thickness is $dy$ (see Figure 11),

$$V = \int_{-\infty}^{\infty} A(y) \, dy$$

Figure 9: Area of annulus or ring, $(2\pi r)dr$.

Figure 10: Slice $A(y)$. 
To compute $A(y)$, note that it is an integral (with respect to $dx$)

$$
A(y) = \int_{-\infty}^{\infty} e^{-r^2} \, dx = \int_{-\infty}^{\infty} e^{-x^2-y^2} \, dx = e^{-y^2} \int_{-\infty}^{\infty} e^{-x^2} \, dx = e^{-y^2} Q
$$

Here, we have used $r^2 = x^2 + y^2$ and

$$
e^{-x^2-y^2} = e^{-x^2} e^{-y^2}
$$

and the fact that $y$ is a constant in the $A(y)$ slice (see Figure 12). In other words,

$$
\int_{-\infty}^{\infty} ce^{-x^2} \, dx = c \int_{-\infty}^{\infty} e^{-x^2} \, dx \quad \text{with} \quad c = e^{-y^2}
$$
It follows that
\[ V = \int_{-\infty}^{\infty} A(y) \, dy = \int_{-\infty}^{\infty} e^{-y^2} Q \, dy = Q \int_{-\infty}^{\infty} e^{-y^2} \, dy = Q^2 \]

Indeed,
\[ Q = \int_{-\infty}^{\infty} e^{-x^2} \, dx = \int_{-\infty}^{\infty} e^{-y^2} \, dy \]
because the name of the variable does not matter. To conclude the calculation read the equation backwards:
\[ \pi = V = Q^2 \implies Q = \sqrt{\pi} \]

We can rewrite \( Q = \sqrt{\pi} \) as
\[ \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} \, dx = 1 \]
An equivalent rescaled version of this formula (replacing \( x \) with \( x/\sqrt{2}\sigma \)) is used:
\[ \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} \, dx = 1 \]

This formula is central to probability and statistics. The probability distribution \( \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2} \) on \(-\infty < x < \infty\) is known as the normal distribution, and \( \sigma > 0 \) is its standard deviation.