Lecture 15: Differentials and Antiderivatives

Differentials

New notation:

\[ dy = f'(x) \, dx \quad (y = f(x)) \]

Both \( dy \) and \( f'(x) \, dx \) are called differentials. You can think of

\[ \frac{dy}{dx} = f'(x) \]

as a quotient of differentials. One way this is used is for linear approximations.

\[ \frac{\Delta y}{\Delta x} \approx \frac{dy}{dx} \]

Example 1. Approximate \( 65^{1/3} \)

Method 1 (review of linear approximation method)

\[
\begin{align*}
  f(x) &= x^{1/3} \\
  f'(x) &= \frac{1}{3}x^{-2/3} \\
  f(x) &\approx f(a) + f'(a)(x - a) \\
  x^{1/3} &\approx a^{1/3} + \frac{1}{3}a^{-2/3}(x - a)
\end{align*}
\]

A good base point is \( a = 64 \), because \( 64^{1/3} = 4 \).

Let \( x = 65 \).

\[
65^{1/3} = 64^{1/3} + \frac{1}{3}64^{-2/3}(65 - 64) = 4 + \frac{1}{3}\left(\frac{1}{16}\right)(1) = 4 + \frac{1}{48} \approx 4.02
\]

Similarly,

\[(64.1)^{1/3} \approx 4 + \frac{1}{480}\]

Method 2 (review)

\[
65^{1/3} = (64 + 1)^{1/3} = [64(1 + \frac{1}{64})]^{1/3} = 64^{1/3}[1 + \frac{1}{64}]^{1/3} = 4\left[1 + \frac{1}{64}\right]^{1/3}
\]

Next, use the approximation \((1 + x)^r \approx 1 + rx\) with \( r = \frac{1}{3} \) and \( x = \frac{1}{64} \).

\[
65^{1/3} \approx 4\left(1 + \frac{1}{3}\left(\frac{1}{64}\right)\right) = 4 + \frac{1}{48}
\]

This is the same result that we got from Method 1.
Method 3 (with differential notation)

\[
y = x^{1/3} \big|_{x=64} = 4
\]
\[
dy = \frac{1}{3} x^{-2/3} dx \big|_{x=64} = \frac{1}{3} \left( \frac{1}{16} \right) dx = \frac{1}{48} dx
\]

We want \( dx = 1 \), since \((x + dx) = 65\). \( dy = \frac{1}{48} \) when \( dx = 1 \).

\[(65)^{1/3} = 4 + \frac{1}{48}\]

What underlies all three of these methods is

\[
y = x^{1/3}
\]
\[
dy \over dx = \frac{1}{3} x^{-2/3} \big|_{x=64}
\]

**Anti-derivatives**

\( F(x) = \int f(x) dx \) means that \( F \) is the antiderivative of \( f \).

Other ways of saying this are:

\( F'(x) = f(x) \) or \( dF = f(x) dx \)

**Examples:**

1. \( \int \sin x \, dx = -\cos x + c \) where \( c \) is any constant.

2. \( \int x^n \, dx = \frac{x^{n+1}}{n+1} + c \) for \( n \neq -1 \).

3. \( \int \frac{dx}{x} = \ln |x| + c \) \hspace{1em} (This takes care of the exceptional case \( n = -1 \) in 2.)

4. \( \int \sec^2 x \, dx = \tan x + c \)

5. \( \int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x + c \) (where \( \sin^{-1} x \) denotes “inverse sin” or arcsin, and not \( \frac{1}{\sin x} \))

6. \( \int \frac{dx}{1 + x^2} = \tan^{-1} (x) + c \)

**Proof of Property 2:** The absolute value \( |x| \) gives the correct answer for both positive and negative \( x \). We will double check this now for the case \( x < 0 \):

\[
\ln |x| = \ln (-x)
\]
\[
\frac{d}{dx} \ln (-x) = \left( \frac{d}{du} \ln (u) \right) \frac{du}{dx} \hspace{1em} \text{where} \hspace{1em} u = -x
\]
\[
\frac{d}{dx} \ln (-x) = \frac{1}{u} (-1) = \frac{1}{-x} (-1) = \frac{1}{x}
\]
Uniqueness of the antiderivative up to an additive constant.

If \( F'(x) = f(x) \), and \( G'(x) = f(x) \), then \( G(x) = F(x) + c \) for some constant factor \( c \).

Proof:
\[
(G - F)' = f - f = 0
\]

Recall that we proved as a corollary of the Mean Value Theorem that if a function has a derivative zero then it is constant. Hence \( G(x) - F(x) = c \) (for some constant \( c \)). That is, \( G(x) = F(x) + c \).

Method of substitution.

Example 1. \( \int x^3(x^4 + 2)^5 dx \)

Substitution:
\[
u = x^4 + 2, \quad du = 4x^3 dx, \quad (x^4 + 2)^5 = u^5, \quad x^3 dx = \frac{1}{4} du
\]

Hence,
\[
\int x^3(x^4 + 2)^5 dx = \frac{1}{4} \int u^5 du = \frac{u^6}{4(6)} = \frac{u^6}{24} + c = \frac{1}{24}(x^4 + 2)^6 + c
\]

Example 2. \( \int \frac{x}{\sqrt{1 + x^2}} dx \)

Another way to find an anti-derivative is “advanced guessing.” First write
\[
\int \frac{x}{\sqrt{1 + x^2}} dx = \int x(1 + x^2)^{-1/2} dx
\]

Guess: \((1 + x^2)^{1/2}\). Check this.
\[
\frac{d}{dx}(1 + x^2)^{1/2} = \frac{1}{2}(1 + x^2)^{-1/2}(2x) = x(1 + x^2)^{-1/2}
\]

Therefore,
\[
\int x(1 + x^2)^{-1/2} dx = (1 + x^2)^{1/2} + c
\]

Example 3. \( \int e^{6x} dx \)

Guess: \( e^{6x} \). Check this:
\[
\frac{d}{dx} e^{6x} = 6e^{6x}
\]

Therefore,
\[
\int e^{6x} dx = \frac{1}{6} e^{6x} + c
\]
Example 4. $\int x e^{-x^2} \, dx$

Guess: $e^{-x^2}$ Again, take the derivative to check:

$$\frac{d}{dx} e^{-x^2} = (-2x)(e^{-x^2})$$

Therefore,

$$\int x e^{-x^2} \, dx = -\frac{1}{2} e^{-x^2} + c$$

Example 5. $\int \sin x \cos x \, dx = \frac{1}{2} \sin^2 x + c$

Another, equally acceptable answer is

$$\int \sin x \cos x \, dx = -\frac{1}{2} \cos^2 x + c$$

This seems like a contradiction, so let’s check our answers:

$$\frac{d}{dx} \sin^2 x = (2 \sin x)(\cos x)$$

and

$$\frac{d}{dx} \cos^2 x = (2 \cos x)(-\sin x)$$

So both of these are correct. Here’s how we resolve this apparent paradox: the difference between the two answers is a constant.

$$\frac{1}{2} \sin^2 x - (-\frac{1}{2} \cos^2 x) = \frac{1}{2} (\sin^2 x + \cos^2 x) = \frac{1}{2}$$

So,

$$\frac{1}{2} \sin^2 x - \frac{1}{2} = \frac{1}{2} (\sin^2 x - 1) = \frac{1}{2} (-\cos^2 x) = -\frac{1}{2} \cos^2 x$$

The two answers are, in fact, equivalent. The constant $c$ is shifted by $\frac{1}{2}$ from one answer to the other.

Example 6. $\int \frac{dx}{x \ln x}$ (We will assume $x > 0$.)

Let $u = \ln x$. This means $du = \frac{1}{x} \, dx$. Substitute these into the integral to get

$$\int \frac{dx}{x \ln x} = \int \frac{1}{u} \, du = \ln u + c = \ln(\ln(x)) + c$$