Lecture 14: Mean Value Theorem and Inequalities

Mean-Value Theorem

The Mean-Value Theorem (MVT) is the underpinning of calculus. It says:

\[
\frac{f(b) - f(a)}{b - a} = f'(c) \quad (\text{for some } c, a < c < b)
\]

Here, \(\frac{f(b) - f(a)}{b - a}\) is the slope of a secant line, while \(f'(c)\) is the slope of a tangent line.

![Figure 1: Illustration of the Mean Value Theorem.](image)

Geometric Proof: Take (dotted) lines parallel to the secant line, as in Fig. 1 and shift them up from below the graph until one of them first touches the graph. Alternatively, one may have to start with a dotted line above the graph and move it down until it touches.

If the function isn’t differentiable, this approach goes wrong. For instance, it breaks down for the function \(f(x) = |x|\). The dotted line always touches the graph first at \(x = 0\), no matter what its slope is, and \(f'(0)\) is undefined (see Fig. 2).
Interpretation of the Mean Value Theorem

You travel from Boston to Chicago (which we’ll assume is a 1,000 mile trip) in exactly 3 hours. At some time in between the two cities, you must have been going at exactly $\frac{1000}{3}$ mph.

$f(t) =$ position, measured as the distance from Boston.

$f(3) = 1000, \quad f(0) = 0, \quad a = 0, \quad$ and $b = 3.$

$$\frac{1000}{3} = \frac{f(b) - f(a)}{b - a} = f'(c)$$

where $f'(c)$ is your speed at some time, $c.$

Versions of the Mean Value Theorem

There is a second way of writing the MVT:

$$f(b) - f(a) = f'(c)(b - a)$$

$$f(b) = f(a) + f'(c)(b - a) \quad \text{for some } c, a < c < b$$

There is also a third way of writing the MVT: change the name of $b$ to $x.$

$$f(x) = f(a) + f'(c)(x - a) \quad \text{for some } c, a < c < x$$

The theorem does not say what $c$ is. It depends on $f, a,$ and $x.$

This version of the MVT should be compared with linear approximation (see Fig. 3).

$$f(x) \approx f(a) + f'(a)(x - a) \quad x \text{ near } a$$
The tangent line in the linear approximation has a definite slope \( f'(a) \). By contrast, the formula is an exact formula. It conceals its lack of specificity in the slope \( f'(c) \), which could be the slope of \( f \) at any point between \( a \) and \( x \).

![Figure 3: MVT vs. Linear Approximation.](image)

**Uses of the Mean Value Theorem.**

**Key conclusions:** (The conclusions from the MVT are theoretical)

1. If \( f'(x) > 0 \), then \( f \) is increasing.
2. If \( f'(x) < 0 \), then \( f \) is decreasing.
3. If \( f'(x) = 0 \) all \( x \), then \( f \) is constant.

**Definition of increasing/decreasing:**
Increasing means \( a < b \Rightarrow f(a) < f(b) \). Decreasing means \( a < b \Rightarrow f(a) < f(b) \).

**Proofs:**

Proof of 1:

\[
\begin{align*}
 a & < b \\
 f(b) & = f(a) + f'(c)(b - a)
\end{align*}
\]

Because \( f'(c) \) and \( (b - a) \) are both positive,

\[
 f(b) = f(a) + f'(c)(b - a) > f(a)
\]

(The proof of 2 is omitted because it is similar to the proof of 1)

Proof of 3:

\[
 f(b) = f(a) + f'(c)(b - a) = f(a) + 0(b - a) = f(a)
\]

Conclusions 1, 2, and 3 seem obvious, but let me persuade you that they are not. Think back to the definition of the derivative. It involves infinitesimals. It’s not a sure thing that these infinitesimals have anything to do with the non-infinitesimal behavior of the function.
Inequalities

The fundamental property $f' > 0 \implies f$ is increasing can be used to deduce many other inequalities.

Example. $e^x$

1. $e^x > 0$
2. $e^x > 1$ for $x > 0$
3. $e^x > 1 + x$

Proofs. We will take property 1 ($e^x > 0$) for granted. Proofs of the other two properties follow:

Proof of 2: Define $f_1(x) = e^x - 1$. Then, $f_1(0) = e^0 - 1 = 0$, and $f'_1(x) = e^x > 0$. (This last assertion is from step 1). Hence, $f_1(x)$ is increasing, so $f(x) > f(0)$ for $x > 0$. That is:

$$e^x > 1 \text{ for } x > 0$$

Proof of 3: Let $f_2(x) = e^x - (1 + x)$.

$$f'_2(x) = e^x - 1 = f_1(x) > 0 \text{ (if } x > 0).$$

Hence, $f_2(x) > 0$ for $x > 0$. In other words,

$$e^x > 1 + x$$

Similarly, $e^x > 1 + x + \frac{x^2}{2}$ (proved using $f_3(x) = e^x - (1 + x + \frac{x^2}{2})$). One can keep on going:

$$e^x > 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} \text{ for } x > 0.$$  

Eventually, it turns out that

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \text{ (an infinite sum)}$$

We will be discussing this when we get to Taylor series near the end of the course.